§5.2 Nonabelian Gauge Theory

Local transformation

Consider an N-component complex scalar field $\varphi(x) = \{\varphi_1(x), \varphi_2(x), \dots, \varphi_N(x)\}$ transforming as

$$\varphi(x) \longmapsto U\varphi(x), \qquad U \in SU(N) \qquad (*)$$

As $\varphi^\dagger = \varphi^\dagger U^\dagger$ and $U^\dagger U = 1$, we get

$$\varphi^\dagger \varphi \longmapsto \varphi^\dagger \varphi$$

$$\partial\varphi^\dagger \partial\varphi \longmapsto \partial\varphi^\dagger \partial\varphi$$

$\rightarrow$ Lagrangian $\mathcal{L} = (\partial_\mu \varphi)^\dagger (\partial^\mu \varphi) - V(\varphi^\dagger \varphi)$ is inv. under $SU(N)$ trf. $(*)$ for any polynomial V.

<u>Question:</u> What happens if we make the trf. U local?

$\rightarrow$ $\varphi^\dagger\varphi$ is still invariant but $\partial\varphi^\dagger\partial\varphi$ is not:

$$\begin{aligned}\partial_\mu \varphi \longmapsto \partial_\mu (U\varphi) &= U\partial_\mu \varphi + (\partial_\mu U)\varphi \\ &= U[\partial_\mu \varphi + (U^\dagger \partial_\mu U)\varphi]\end{aligned}$$

→ introduce covariant derivative D_μ:

$$D_\mu \varphi(x) = \partial_\mu \varphi(x) - i A_\mu(x) \varphi(x) \qquad (1)$$

with $A_\mu \mapsto U A_\mu U^\dagger - i(\partial_\mu U) U^\dagger$

$$= U A_\mu U^\dagger + i U \partial_\mu U^\dagger \qquad (2)$$

check: $D_\mu \varphi \mapsto U\,[\partial_\mu \varphi + (U^\dagger \partial_\mu U)$
$\quad - i\,(A_\mu U^\dagger + i \partial_\mu U^\dagger) U]\,\varphi$

$\quad U^\dagger U = \mathbb{1} \to (\partial_\mu U^\dagger) U + U^\dagger \partial_\mu U = 0$
$\quad \Leftrightarrow (\partial_\mu U^\dagger) U = - U^\dagger \partial_\mu U$

$\quad \overset{!}{=} U D_\mu \varphi$

→ φ is section of non-Abelian bundle with connection A_μ over space time

<u>Remarks</u>:

1) A_μ are valued in the Lie algebra of $SU(N)$ → hermitian
for $SU(2)$: $U = e^{i \vec{\theta} \cdot \frac{\vec{\sigma}}{2}}$, where
$\vec{\theta} \cdot \vec{\sigma} = \theta^a \sigma^a$,
σ^a Pauli matrices

2) Writing $U = e^{i\vec{\theta}\cdot\vec{T}}$ with T^a generators of $SU(N)$, we have

$$A_\mu \longmapsto A_\mu + i\theta^a [T^a, A_\mu] + \partial_\mu \theta^a T^a$$

under an infinitesimal trf. (3)

$$U \simeq \mathbb{1} + i\vec{\theta}\cdot\vec{T}$$

3) taking the trace of (3), we see that

$\mathrm{tr}\, \delta A_\mu = 0 \rightarrow \mathrm{tr} A_\mu$ is fixed under (3)

$\rightarrow$ take A_μ to be traceless and hermitian

4) We have

$$[T^a, T^b] = i f^{abc} T^c$$

structure constants (e.g. $f^{abc} = \varepsilon^{abc}$ for $SU(2)$)

$\rightarrow$ (3) can be written as

$$A_\mu^a \longmapsto A_\mu^a - f^{abc}\theta^b A_\mu^c + \partial_\mu \theta^a$$

$\rightarrow$ locally, A_μ^a's transform in adjoint rep. of gauge group

5) A_μ is known as non-Abelian gauge potential

A Lagrangian invariant under (2) is called "gauge invariant".

Construction of the field strength

We now consider the gauge invariant Lagrangian

$$\mathcal{L} = (D_\mu \varphi)^\dagger (D_\mu \varphi) - V(\varphi^\dagger \varphi)$$

In the abelian $U(1)$-case, we also had Maxwell term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$\rightarrow$ rewrite everything in diff. forms

$$\mathcal{A} = A_\mu dx^\mu$$

$$\rightarrow \mathcal{A} \wedge \mathcal{A} = A_\mu A_\nu \, dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2} [A_\mu, A_\nu] dx^\mu \wedge dx^\nu$$

$$(2): \quad \mathcal{A} \longmapsto U \mathcal{A} U^\dagger + U d U^\dagger$$

0-form, $dU^\dagger = \partial_\mu U^\dagger dx^\mu$

$$d\mathcal{A} \longmapsto U\, d\mathcal{A}\, U^\dagger + \underline{dU\, \mathcal{A}\, U^\dagger} - \underline{U \mathcal{A}\, dU^\dagger} + \underline{dU\, dU^\dagger} \tag{4}$$

and

$$\mathcal{A}^2 \longmapsto U \mathcal{A}^2 U^\dagger + U \mathcal{A}\, dU^\dagger + U dU^\dagger U \mathcal{A} U^\dagger + U dU^\dagger U dU^\dagger$$

using $U dU^\dagger = - dU U^\dagger$ gives

$$= U \mathcal{A}^2 U^\dagger + \underline{U \mathcal{A}\, dU^\dagger} - \underline{dU\, \mathcal{A}\, U^\dagger} - \underline{dU\, dU^\dagger} \tag{5}$$

we see that (4) + (5) gives:

$$d\mathcal{A} + \mathcal{A}^2 \longmapsto U (d\mathcal{A} + \mathcal{A}^2) U^\dagger$$

$\rightarrow$ define field strength

$$\mathcal{F} := d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

and $\mathcal{F} \longmapsto U \mathcal{F} U^\dagger$ (6)

In components we have

$$\mathcal{F} = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

where

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$
$$= \underbrace{\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_{\mu b} A_{\nu c}\right)}_{=: F^a_{\mu\nu}} T_a$$

The connection one-form $\mathcal{A}$ is again related to the physical field via

$$\mathcal{A} = -i A^P_\mu dx^\mu$$

$$\text{so} \quad F_{\mu\nu} = -i F^P_{\mu\nu}$$

Omitting the superscript P, we write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]$$

Yang-Mills Lagrangian

The trf. property (6) immediately gives the correct gauge inv. Maxwell-term for non-Abelian gauge potentials:

$$\mathcal{L} = -\frac{1}{2g^2} \, tr\, F_{\mu\nu} F^{\mu\nu} \qquad (7)$$

→ the normalization $tr\, T^a T^b = \frac{1}{2} \delta^{ab}$ gives $\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}$

"pure Yang-Mills theory"

Variation with respect to A_μ gives

$$D_\mu F^{\mu\nu} = 0 \quad \text{or} \quad D * \mathcal{F} = 0 \qquad (8)$$

Interactions

The Lagrangian (7) contains

a) quadratic term

$$(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2$$

b) cubic term

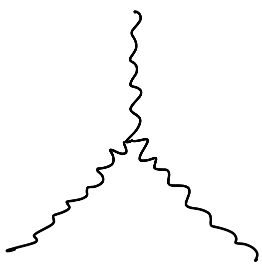
$$f^{abc} A^{b\mu} A^{c\nu} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)$$

c) and a quartic term

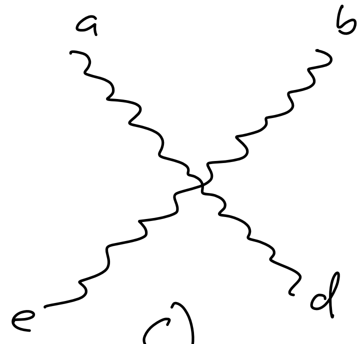
$$(f^{abc} A^b_\mu A^c_\nu)^2$$

→ corresponding Feynman rules

a) : 



b)



c)

→ interaction couplings f^{abc} are completely determined by group theory, renormalization does not change their relative strength

't Hoof double-line formalism

propagator has the form

$$\langle 0| T[A_\mu(x)^i{}_j \, A_\nu(0)^k{}_l]|0\rangle$$

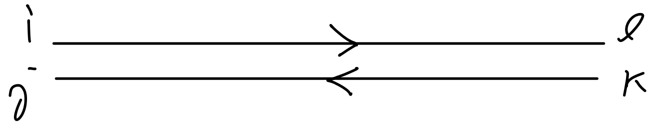
$$= \langle 0|T[A^a_\mu(x) \, A_\nu{}^b(0)]|0\rangle (T^a)^i{}_j \, (T^b)^k{}_l$$

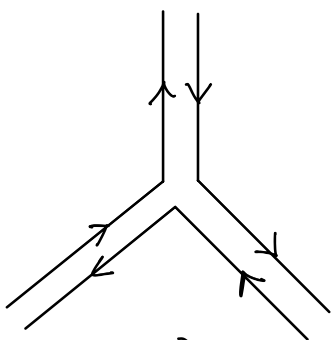
$$\sim \delta^{ab} (T^a)^i{}_j \, (T^b)^k{}_l \sim \delta^i_l \, \delta^k_j \quad \text{"Casimir"}$$

fundamental rep.

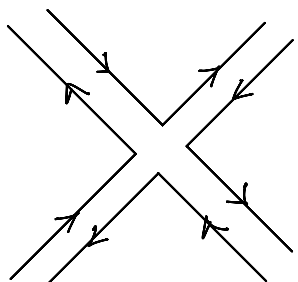
matrix structure $A^i_{\mu j}$ suggests (following 't Hooft) to introduce "double-line" formalism

a)





b)



c)

Coupling to matter fields

Let φ be a scalar field in rep. R of gauge group G

$\longrightarrow$ covariant derivative:

$$D_\mu \varphi = (\partial_\mu - i A_\mu^a T_R^a)\varphi$$

ath generator in rep. R

Similarly, we can couple A_μ to a fermion field via

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$$= \bar{\psi}(i\gamma^\mu \partial_\mu + \gamma^\mu A_\mu^a T_R^a - m)\psi$$

in rep. R